Introduction to Mathematics and Modeling

lecture 6

Complex numbers

academic year : 17-18
lecture : 6
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This week

The Riemann zeta-function

1. Appendix A.7: complex numbers
2. Application: impedance
A complex number is a vector in $\mathbb{R}^2$. 

- The complex plane
- Appendix A.7
- 1.1
- University of Twente
- Introduction to Mathematics and Modeling
- Lecture 6: Complex numbers
- 2/40
A complex number is a vector in $\mathbb{R}^2$.
In stead of $\mathbb{R}^2$ we write $\mathbb{C}$. 
A complex number is a vector in $\mathbb{R}^2$.

In stead of $\mathbb{R}^2$ we write $\mathbb{C}$.

Rather than $x$- and $y$-axis, we call them the real axis and imaginary axis.
A complex number is a vector in $\mathbb{R}^2$.

In stead of $\mathbb{R}^2$ we write $\mathbb{C}$.

Rather than $x$- and $y$-axis, we call them the real axis and imaginary axis.

The complex number $i$ is defined as $(0, 1)$. 
Addition is defined termwise: if $z = (x, y)$ and $w = (u, v)$, then

$$z + w = (x + y, y + v)$$

Scalar multiplication is defined termwise: if $z = (x, y)$ and $\alpha \in \mathbb{R}$, then

$$\alpha z = (\alpha x, \alpha y)$$

Notebook: Sum.nb
Definition

Let \( z = (x, y) \) and \( w = (u, v) \) be two complex numbers. The \textbf{product of} \( z \) \textbf{and} \( w \) is defined as

\[
z w = (x u - y v, x v + y u)\]

\[
(1, 2)(4, -1) = (1 \cdot 4 - 2 \cdot (-1), 1 \cdot (-1) + 2 \cdot 4) = (6, 7).
\]

\[
(2, 0)(3, -4) = (2 \cdot 3 - 0 \cdot (-4), 2 \cdot (-4) + 0 \cdot 3) = (2 \cdot 3, 2 \cdot (-4)) = (6, -8).
\]

\[
i^2 = i \cdot i = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0).
\]
### Definition

Let \( z = (x, y) \) and \( w = (u, v) \) be two complex numbers. The **product of** \( z \) **and** \( w \) is defined as

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**Examples:**

- \((1, 2)(4, -1) = (1 \cdot 4 - 2(-1), 1(-1) + 2 \cdot 4) = (6, 7)\).
**Definition**

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- \((1, 2)(4, -1) = (1 \cdot 4 - 2(-1), 1(-1) + 2 \cdot 4) = (6, 7)\).
- \((2, 0)(3, -4) = (2 \cdot 3 - 0(-4), 2(-4) + 0 \cdot 3) = (2 \cdot 3, 2(-4)) = 2(3, -4)\).
Definition

Let \( z = (x, y) \) and \( w = (u, v) \) be two complex numbers. The **product of** \( z \) **and** \( w \) is defined as

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- \(i^2 = i \cdot i = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)\).

**Notebook:** Product.nb
**Definition**

Let $z = (x, y)$ and $w = (u, v)$ be two complex numbers. The **product of** $z$ **and** $w$ **is defined as**

$$zw = (xu - yv, xv + yu).$$

**Examples:**

- $(1, 2)(4, -1) = (1 \cdot 4 - 2(-1), 1(-1) + 2 \cdot 4) = (6, 7)$.
- $(2, 0)(3, -4) = (2 \cdot 3 - 0(-4), 2(-4) + 0 \cdot 3) = (2 \cdot 3, 2(-4)) = 2(3, -4)$.
- $i^2 = i \cdot i = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)$.

**Exercise**

Let $z = (2, 1)$ and $w = (3, 1)$, calculate $z^2$, $zw$, and $w^2$.

**Notebook:** Product.nb
Definition

Let \( z = (x, y) \) and \( w = (u, v) \) be two complex numbers. The **product of** \( z \) **and** \( w \) is defined as

\[
z w = (x u - y v, x v + y u)\]

**Examples:**

\[
\begin{align*}
(1, 2)(4, -1) &= (1 \cdot 4 - 2 \cdot (-1), 1 \cdot (-1) + 2 \cdot 4) = (6, 7) \\
(2, 0)(3, -4) &= (2 \cdot 3 - 0 \cdot (-4), 2 \cdot (-4) + 0 \cdot 3) = (6, -8)
\end{align*}
\]

\[
i^2 = i \cdot i = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)
\]

**Exercise**

Let \( z = (2, 1) \) and \( w = (3, 1) \), calculate \( z^2, zw, \) and \( w^2 \).

**Answer**

\[
\begin{align*}
z^2 &= (3, 4) \\
zw &= (5, 5) \\
w^2 &= (8, 6)
\end{align*}
\]
Convention

Every real number \( x \) is identified with the complex number \((x, 0)\).
The real axis

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Every real number \( x \) is identified with the complex number \( (x, 0) \).

Examples: \( 0 = (0, 0) \), \( 1 = (1, 0) \), \( -1 = (-1, 0) \).
The real axis

Convention

Every real number $x$ is identified with the complex number $(x, 0)$.

Examples: $0 = (0, 0)$, $1 = (1, 0)$, $-1 = (-1, 0)$.

The complex numbers on the real axis behave just like the real numbers in $\mathbb{R}$:

- $x + y \rightarrow (x, 0) + (y, 0) = (x + y, 0 + 0) = (x + y, 0)$.
- $x - y \rightarrow (x, 0) - (y, 0) = (x - y, 0 - 0) = (x - y, 0)$.
- $xy \rightarrow (x, 0)(y, 0) = (xy - 0 \cdot 0, x \cdot 0 + 0 \cdot y) = (xy, 0)$. 
By identifying $x \in \mathbb{R}$ with the complex number $(x, 0)$, we regard the points on the real axis as the real number line.
Real numbers are complex numbers

- By identifying \( x \in \mathbb{R} \) with the complex number \((x, 0)\), we regard the points on the real axis as the real number line.

- \( i^2 = -1 \)
Real numbers are complex numbers

- By identifying $x \in \mathbb{R}$ with the complex number $(x, 0)$, we regard the points on the real axis as the real number line.

- $i^2 = -1$

- The complex numbers are an expansion of the real numbers:
Let $z$, $w$ and $u$ be complex numbers. Define $z - w$ and $-z$ in the usual way, then

1. $z + w = w + z$
2. $z + w + u = z + (w + u) = (z + w) + u$
3. $z + 0 = z$
4. $-z = (-1)z$
5. $z - w = z + (-w)$
6. $z - z = 0$
7. $zw = wz$
8. $z \cdot 1 = z$
9. $z \cdot 0 = 0$
10. $zwu = z(wu) = (zw)u$
11. $z(w + u) = zw + zu$
12. $z(w - u) = zw - zu$
The canonical form

Theorem

Let $z = (x, y)$ be a complex number, with $x$ and $y$ real. Then

$$z = x + i y.$$
The canonical form

**Theorem**

Let \( z = (x, y) \) be a complex number, with \( x \) and \( y \) real. Then

\[
z = x + i y.
\]

**Proof:**

\[
x + i y = (x, 0) + (0, 1)(y, 0)
\]
The canonical form

Theorem

Let \( z = (x, y) \) be a complex number, with \( x \) and \( y \) real. Then

\[ z = x + i y. \]

Proof:

\[
x + i y = (x, 0) + (0, 1)(y, 0) \\
= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y)
\]
The canonical form

**Theorem**

Let $z = (x, y)$ be a complex number, with $x$ and $y$ real. Then

$$z = x + i y.$$  

**Proof:**

$$x + i y = (x, 0) + (0, 1)(y, 0)$$

$$= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y)$$

$$= (x, 0) + (0, y) = (x, y) = z.$$
The canonical form

**Theorem**

Let \( z = (x, y) \) be a complex number, with \( x \) and \( y \) real. Then

\[
z = x + i y.\]

**Proof:**

\[
x + i y = (x, 0) + (0, 1)(y, 0)
= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y)
= (x, 0) + (0, y) = (x, y) = z.
\]

**Definition**

The form \( x + i y \) is called the **canonical form** of \( z \).
The canonical form

**Theorem**

Let \( z = (x, y) \) be a complex number, with \( x \) and \( y \) real. Then

\[
z = x + i \, y.
\]

**Proof:**

\[
x + i \, y = (x, 0) + (0, 1)(y, 0)
= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y)
= (x, 0) + (0, y) = (x, y) = z.
\]

**Definition**

The form \( x + i \, y \) is called the **canonical form** of \( z \).

**Henceforth we will always write complex numbers in canonical form.**
Let $z = x + iy$ and $w = u + iv$ be two complex numbers, with $x, y, u$ and $v$ real. Then

$$z + w = (x + iy) + (u + iv)$$
$$= x + u + iy + iv$$
$$= (x + u) + i(y + v).$$
Let $z = x + iy$ and $w = u + iv$ be two complex numbers, with $x$, $y$, $u$ and $v$ real. Then

$$z + w = (x + iy) + (u + iv)$$
$$= x + u + iy + iv$$
$$= (x + u) + i(y + v).$$

For the product of $z$ and $w$ we have

$$zw = (x + iy)(u + iv).$$
Let \( z = x + i\, y \) and \( w = u + i\, v \) be two complex numbers, with \( x, \, y, \, u \) and \( v \) real. Then

\[
z + w = (x + i\, y) + (u + i\, v) = x + u + i\, y + i\, v = (x + u) + i(y + v)\.
\]

For the product of \( z \) and \( w \) we have

\[
z\, w = (x + i\, y)(u + i\, v) = xu + (i\, y)(i\, v) + x(i\, v) + (i\, y)u
\]
Let \( z = x + i \, y \) and \( w = u + i \, v \) be two complex numbers, with \( x, \, y, \, u \) and \( v \) real. Then

\[
z + w = (x + i \, y) + (u + i \, v) = x + u + i \, y + i \, v = (x + u) + i(y + v).
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For the product of \( z \) and \( w \) we have

\[
z \, w = (x + i \, y)(u + i \, v) = x \, u + (i \, y)(i \, v) + x(i \, v) + (i \, y)u = x \, u + i^2 \, y \, v + i \, x \, v + i \, y \, u
\]
Let \( z = x + i\,y \) and \( w = u + i\,v \) be two complex numbers, with \( x, y, u \) and \( v \) real. Then

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For the product of \( z \) and \( w \) we have

\[
z\,w = (x + i\,y)(u + i\,v) = xu + (i\,y)(i\,v) + x(i\,v) + (i\,y)u = xu + i^2\,y\,v + ix\,v + iy\,u = (xu - y\,v) + i(x\,v + y\,u).
\]
Let \( z = x + i y \) and \( w = u + i v \) be two complex numbers, with \( x, y, u \) and \( v \) real. Then

\[
z + w = (x + i y) + (u + i v)
\]

**Exercise**

Let \( z = 2 + i \) and \( w = 3 + i \), calculate \( z^2 \), \( zw \), and \( w^2 \).

\[
zw = (x + iy)(u + iv)
\]

\[
= xu + (iy)(iv) + x(iv) + (iy)u
\]

\[
= xu + i^2 yv + ixv + iyu
\]

\[
= (xu - yv) + i(xv + yu).
\]
Let \( z = x + i y \) and \( w = u + i v \) be two complex numbers, with \( x, y, u \) and \( v \) real. Then

\[
 z + w = (x + i y) + (u + i v)
\]

**Exercise**

Let \( z = 2 + i \) and \( w = 3 + i \), calculate \( z^2 \), \( zw \), and \( w^2 \).

**Answer**

\[
 z^2 = 3 + 4i \\
 zw = 5 + 5i \\
 w^2 = 8 + 6i
\]
**Definition**

Let \( z = x + i \, y \) be a complex number with \( x \) and \( y \) real. Then \( x \) is the **real part of** \( z \) and \( y \) is the **imaginary part of** \( z \). We denote

\[
x = \text{Re} \, z \quad \text{and} \quad y = \text{Im} \, z.
\]
**Definition**

Let $z = x + iy$ be a complex number with $x$ and $y$ real. Then the **conjugate of $z$ is the complex number** $\bar{z}$ defined by

$$\bar{z} = x - iy.$$ 

- The conjugate of $z$ is the reflection of $z$ across the real axis.

![Diagram showing the conjugate of a complex number](image)

**Notebook:** Conjugate.nb
**Definition**

Let $z = x + iy$ be a complex number with $x$ and $y$ real. Then the absolute value of $z$ is the distance of $z$ to 0:

$$|z| = \sqrt{x^2 + y^2}.$$  

- The definition is based on the Pythagorean theorem.
- The absolute value is sometimes called **modulus** or **norm**.
Properties of conjugation and absolute value

Let $z$ and $w$ be complex numbers, then

1. $z + w = \bar{z} + \bar{w}$
2. $z - w = \bar{z} - \bar{w}$
3. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
4. $|z|^2 = z \overline{z}$
5. $|z \cdot w| = |z| \cdot |w|$
6. $|z + w| \leq |z| + |w|$

Property 6 is called the triangular inequality.
Let $z$ and $w$ be complex numbers, then

1. $z + w = \bar{z} + \bar{w}$
2. $z - w = \bar{z} - \bar{w}$
3. $\bar{z}w = \bar{z}\bar{w}$
4. $|z|^2 = z\bar{z}$
5. $|zw| = |z||w|$
6. $|z + w| \leq |z| + |w|$

Property 6 is called the **triangular inequality**.
Properties of conjugation and absolute value

Let \( z \) and \( w \) be complex numbers, then

1. \( \overline{z + w} = \overline{z} + \overline{w} \)

2. \( \overline{z - w} = \overline{z} - \overline{w} \)

3. \( z \cdot w = \overline{z} \cdot \overline{w} \)

4. \( |z|^2 = z \overline{z} \)

5. \( |z \cdot w| = |z| \cdot |w| \)

Exercise

Let \( z = 2 + i \), calculate \( |z| \), \( |z|^2 \) and \( |z^2| \).
Properties of conjugation and absolute value

- Let $z$ and $w$ be complex numbers, then

  1. $\overline{z + w} = \overline{z} + \overline{w}$
  2. $\overline{z - w} = \overline{z} - \overline{w}$
  3. $zw = \overline{z} \overline{w}$
  4. $|z|^2 = z \overline{z}$
  5. $|zw| = |z||w|

Exercise

Let $z = 2 + i$, calculate $|z|$, $|z|^2$ and $|z^2|$.

Answer

$|z| = \sqrt{5}$

$|z|^2 = |z^2| = 5$  ($|z|^2 = |z^2|$ follows from 5.)
The real and imaginary part

**Theorem**

For every complex number \( z \) the following holds:

(1) \(
\text{Re } z = \frac{\bar{z} + z}{2}
\)

(2) \(
\text{Im } z = \frac{\bar{z} - z}{2}i
\)

Write \( z = x + iy \), then

\[
\bar{z} + z = (x + iy) + (x - iy) = 2x
\]

and

\[
\bar{z} - z = (x + iy) - (x - iy) = 2iy
\]

\[
\frac{\bar{z} - z}{2}i = -\frac{1}{2}i - \frac{1}{2}i(z - \bar{z}) = y = \text{Im } z.
\]
The real and imaginary part

**Theorem**

For every complex number $z$ the following holds:

1. $\text{Re } z = \frac{\overline{z} + z}{2}$
2. $\text{Im } z = \frac{\overline{z} - z}{2}i$

- Write $z = x + iy$, then

$$z + \overline{z} = (x + iy) + (x - iy)$$
The real and imaginary part

**Theorem**

For every complex number $z$ the following holds:

1. $\text{Re } z = \frac{\bar{z} + z}{2}$
2. $\text{Im } z = \frac{\bar{z} - z}{2}i$

- Write $z = x + iy$, then

\[
z + \bar{z} = (x + iy) + (x - iy) = 2x = 2 \text{Re } z,
\]
The real and imaginary part

Theorem

For every complex number $z$ the following holds:

1. $\text{Re } z = \frac{\bar{z} + z}{2}$

2. $\text{Im } z = \frac{\bar{z} - z}{2} i$

- Write $z = x + iy$, then

\[
z + \bar{z} = (x + iy) + (x - iy) = 2x = 2 \text{ Re } z,
\]

and

\[
z - \bar{z} = (x + iy) - (x - iy)
\]
The real and imaginary part

Theorem

For every complex number $z$ the following holds:

(1) $\text{Re } z = \frac{\bar{z} + z}{2}$

(2) $\text{Im } z = \frac{\bar{z} - z}{2} i$

Write $z = x + iy$, then

$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2 \text{ Re } z,$$

and

$$z - \bar{z} = (x + iy) - (x - iy) = 2iy,$$
**Theorem**

For every complex number $z$ the following holds:

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Write $z = x + iy$, then

$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2 \text{Re } z,$$

and

$$z - \bar{z} = (x + iy) - (x - iy) = 2iy,$$

$$-\frac{1}{2} i (z - \bar{z}) = y = \text{Im } z.$$
Assignment: IMM2 - Tutorial 6.1
Problem

For arbitrary \( z \neq 0 \), find a complex number \( w \) such that \( zw = 1 \).
Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$.

- Assume that $zw = 1$, then

$$\bar{z} \cdot \bar{w} = \bar{z}$$
Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$.

- Assume that $zw = 1$, then

$$\overline{z} zw = \overline{z} \quad \Rightarrow \quad |z|^2 w = \overline{z}$$
Problem

For arbitrary \( z \neq 0 \), find a complex number \( w \) such that \( zw = 1 \).

- Assume that \( zw = 1 \), then

\[
\bar{z} \bar{w} = \bar{z} \quad \Rightarrow \quad |z|^2 w = \bar{z} \quad \Rightarrow \quad w = \frac{1}{|z|^2} \bar{z}
\]
Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$.

- Assume that $zw = 1$, then

\[
\overline{z} z w = \overline{z} \quad \Rightarrow \quad |z|^2 w = \overline{z} \quad \Rightarrow \quad \frac{1}{z} = w = \frac{1}{|z|^2} \overline{z}
\]

- The number $w$ is called the \textbf{reciproke of} $z$ and is denoted as $\frac{1}{z}$.
Problem

For arbitrary \( z \neq 0 \), find a complex number \( w \) such that \( zw = 1 \).

- Assume that \( zw = 1 \), then

\[
\overline{z} z w = \overline{z} \quad \Rightarrow \quad |z|^2 w = \overline{z} \quad \Rightarrow \quad \frac{1}{z} = w = \frac{1}{|z|^2} \overline{z}
\]

- The number \( w \) is called the **reciproke** of \( z \) and is denoted as \( \frac{1}{z} \).

- The reciproke of \( z \) is sometimes denoted as \( z^{-1} \).
Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$.

- Assume that $zw = 1$, then

$$\overline{z}zw = \overline{z} \quad \Rightarrow \quad |z|^2 w = \overline{z} \quad \Rightarrow \quad \frac{1}{z} = w = \frac{1}{|z|^2} \overline{z}$$

- The number $w$ is called the **reciproce** of $z$ and is denoted as $\frac{1}{z}$.

- The reciprocal of $z$ is sometimes denoted as $z^{-1}$.

- If $z = x + iy$ with $x$ and $y$ real, then

$$\frac{1}{z} = \frac{1}{|z|^2} \overline{z} = \frac{1}{x^2 + y^2} (x - iy) = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i.$$
Definition

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$\frac{u}{z} = u \cdot \frac{1}{z}.$$

Notebook: Quotient.nb
**Definition**

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$ \frac{u}{z} = u \cdot \frac{1}{z}. $$

Equivalently we can write

$$ \frac{u}{z} = \frac{1}{|z|^2} u \overline{z}. $$

Notebook: Quotient.nb
**Definition**

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$\frac{u}{z} = u \frac{1}{z}.$$

- Equivalently we can write $\frac{u}{z} = \frac{1}{|z|^2} u \overline{z}$.
- Practical approach: multiply numerator and denominator with $\overline{z}$:

$$\frac{u}{z} = \frac{u \overline{z}}{z \overline{z}},$$

and elaborate $u \overline{z}$. 

- Notebook: Quotient.nb
Definition

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$\frac{u}{z} = u \cdot \frac{1}{z}.$$ 

Equivalently we can write

$$\frac{u}{z} = \frac{1}{|z|^2} u \overline{z}.$$ 

Practical approach: multiply numerator and denominator with $\overline{z}$:

$$\frac{u}{z} = \frac{u \overline{z}}{z \overline{z}},$$

and elaborate $u \overline{z}$.

Example:

$$\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)}.$$
Definition

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$\frac{u}{z} = \frac{1}{z}.$$

 equivalently we can write $\frac{u}{z} = \frac{1}{|z|^2} u \overline{z}$.

Practical approach: multiply numerator and denominator with $\overline{z}$:

$$\frac{u}{z} = \frac{u \overline{z}}{z \overline{z}},$$

and elaborate $u \overline{z}$.

Example:

$$\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{1^2 + 2^2}.$$
**Definition**

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$\frac{u}{z} = u \frac{1}{\bar{z}}.$$  

Notebook: Quotient.nb

- Equivalently we can write \[ \frac{u}{z} = \frac{1}{|z|^2} u \bar{z}. \]
- Practical approach: multiply numerator and denominator with $\bar{z}$:

$$\frac{u}{z} = \frac{u \bar{z}}{z \bar{z}},$$

and elaborate $u \bar{z}$.

- Example:

$$\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{1^2 + 2^2} = \frac{5 - 5i}{5} = 1 - i.$$
Definition

Let \( z \) and \( w \) be complex numbers. If \( z \neq 0 \) then the **quotient of** \( u \) and \( z \) is defined as the product of \( u \) and the reciprocal of \( z \):

\[
\frac{u}{z} = u \frac{1}{z}
\]

Notebook: Quotient.nb

Equivalently we can write

\[
\frac{u}{z} = \frac{1}{|z|^2} u z.
\]

Practical approach: multiply numerator and denominator with \( \overline{z} \):

\[
\frac{u}{z} = \frac{u \overline{z}}{z \overline{z}},
\]

and elaborate \( u \overline{z} \).

Example:

\[
\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{5} = 1 - i.
\]

Exercise

Let \( z = 3 + i \) and \( w = 3 - 4i \), calculate \( z/w \) and \( w/z \).
**Definition**

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the **quotient of $u$ and $z$** is defined as the product of $u$ and the reciprocal of $z$:

$$u/z = u \cdot \frac{1}{z}.$$ 

Equivalently we can write $u/z = \frac{1}{|z|^2} u z$.

**Exercise**

Let $z = 3 + i$ and $w = 3 - 4i$, calculate $z/w$ and $w/z$.

**Answer**

$$\frac{z}{w} = \frac{1}{5} + \frac{3}{5}i,$$

$$\frac{w}{z} = \frac{1}{2} - \frac{3}{2}i.$$
Let $u \neq 0$, $v$, $z \neq 0$ and $w$ be complex numbers.

1. $\frac{w}{1} = w$
2. $\frac{w}{z} = \overline{\frac{w}{z}}$
3. $\frac{1}{w/z} = \frac{w}{z}$ (for $w \neq 0$)
4. $\frac{w}{z} = \frac{w}{\overline{z}}$
5. $|w/z| = \frac{|w|}{|z|}$

For all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ the following holds:

1. $z^m z^n = z^{m+n}$
2. $(z^m)^n = z^{mn}$
3. $\frac{1}{z^m} = z^{-m}$
4. $z^n w^n = (zw)^n$
5. $\left(\frac{w}{z}\right)^n = \frac{w^n}{z^n}$
Assignment: IMM2 - Tutorial 6.2
Definition

*The argument* of a complex number $z \neq 0$ is the angle that the line through 0 and $z$ makes with the positive real axis.

*The argument of $z$ is denoted as* $\arg(z)$.

- The argument of 0 is not defined.
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The argument

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- The argument is measured from the positive real axis.
- If the direction is counter-clockwise, the argument is positive.
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If the direction is counter-clockwise, the argument is positive.

If the direction is clockwise, the argument is negative.

The argument is determined up to a multiple of $2\pi$.

**Exercise**

Find $\arg(z)$ for $z = 1$, $z = i$, $z = -1$ and $z = -i$.

**Answer**

$\arg(1) = 0$

$\arg(i) = \frac{\pi}{2}$

$\arg(-1) = \pi$

$\arg(-i) = -\frac{\pi}{2}$ or $\frac{3\pi}{2}$
Definition

The Euler function is the function that assigns to every real number \( \varphi \) the complex number

\[
e^{i\varphi} = \cos \varphi + i \sin \varphi.
\]
Definition

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- The real part of $e^{i\varphi}$ is $\cos \varphi$, the imaginary part of $e^{i\varphi}$ is $\sin \varphi$.
- The complex number $e^{i\varphi}$ is the number on the unit circle with argument $\varphi$. 

UNIVERSITY OF TWENTE

Introduction to Mathematics and Modeling

Lecture 6: Complex numbers
The Euler function

**Theorem**

For every real number \( \varphi \) and \( \psi \) we have

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e^{i(\varphi + \psi)} = e^{i\varphi} e^{i\psi}
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- Use trigonometry formulas to derive

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e^{i(\varphi + \psi)} = \cos(\varphi + \psi) + i \sin(\varphi + \psi)
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\[
= \cos \varphi \cos \psi - \sin \varphi \sin \psi + i(\sin \varphi \cos \psi + \cos \varphi \sin \psi).
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The Euler function

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- Expand the right-hand side:

\[
e^{i\varphi} e^{i\psi} = (\cos \varphi + i \sin \varphi)(\cos \psi + i \sin \psi)
= \cos \varphi \cos \psi + i^2 \sin \varphi \sin \psi + i \sin \varphi \cos \psi + i \cos \varphi \sin \psi
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The Euler function

3.3

Theorem

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\[
= \cos \varphi \cos \psi - \sin \varphi \sin \psi + i(\sin \varphi \cos \psi + \cos \varphi \sin \psi)
\]

\[
= e^{i(\varphi + \psi)}.
\]
1. $e^{i0} = 1$

2. $|e^{i\phi}| = 1$

3. $e^{i(\phi + \psi)} = e^{i\phi} e^{i\psi}$

4. $(e^{i\phi})^n = e^{in\phi}$ for all $n \in \mathbb{Z}$.

5. $\overline{e^{i\phi}} = e^{-i\phi} = \frac{1}{e^{i\phi}}$

*The Simpsons* (from episode ‘Treehouse of Horror VI’)

*The Euler function Cheat Sheet*
Theorem

Every complex number $z \neq 0$ can be written as the product of a positive real number and an Euler function value. In particular, if $r = |z|$ and $\varphi = \arg z$, then

$$z = r \ e^{i\varphi}$$
Theorem

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$$z = r e^{i \varphi}$$

- Write $z = x + i y$ with $x$ and $y$ real, then

$$\cos \varphi = \frac{x}{r} \quad \text{and} \quad \sin \varphi = \frac{y}{r}.$$
Theorem

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Theorem

Let $z$ and $w$ be two complex numbers written in polar coordinates:

$$z = r \, e^{i\varphi} \quad \text{and} \quad w = s \, e^{i\psi},$$

then

$$z \, w = r s e^{i(\varphi+\psi)} \quad \text{and (if } w \neq 0) \quad \frac{z}{w} = \frac{r}{s} e^{i(\varphi-\psi)}.$$
Theorem

Let $z$ and $w$ be two complex numbers written in polar coordinates:

$$z = r e^{i\varphi} \quad \text{and} \quad w = s e^{i\psi},$$

then

$$zw = rse^{i(\varphi+\psi)} \quad \text{and (if } w \neq 0) \quad \frac{z}{w} = \frac{r}{s} e^{i(\varphi-\psi)}.$$

In other words:

- the absolute value of $zw$ is the product of $|z|$ and $|w|$, 
- the argument of $zw$ is the sum of $\arg z$ and $\arg w$, 

and:

- the absolute value of $z/w$ is the quotient of $|z|$ and $|w|$, 
- the argument of $z/w$ is the difference of $\arg z$ and $\arg w$. 

Corollary

Let $w = r e^{i\varphi}$. Then multiplication of an arbitrary complex number $z$ with $w$ can be constructed geometrically by scaling $z$ with scale factor $r$, and by rotating $z$ over an angle $\varphi$ about 0.
Corollary

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**Example:** let \( w = \sqrt{3} + i = 2e^{i\pi/6} \), then \( zw \) is obtained by scaling \( z \) with factor 2, and by rotating \( z \) over an angle of 30°.
Corollary

Let \( w = r e^{i\phi} \). Then multiplication of an arbitrary complex number \( z \) with \( w \) can be constructed geometrically by scaling \( z \) with scale factor \( r \), and by rotating \( z \) over an angle \( \phi \) about 0.

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**Exercise**

Let \( S \) be the square with vertices 0, 1, \( i \) and \( 1+i \). Draw the figure \( T \) obtained by multiplying all points of \( S \) with \( w = 1+i \).
Corollary

Let \( w = r e^{i\phi} \). Then multiplication of an arbitrary complex number \( z \) with \( w \) can be constructed geometrically by scaling \( z \) with scale factor \( r \), and by rotating \( z \) over an angle \( \phi \) about \( 0 \).

Example: let \( w = \sqrt{3} + i = 2e^{i\pi/6} \), then \( zw \) is obtained by scaling \( z \) with factor \( 2 \), and by rotating \( z \) over an angle of \( 30^\circ \).

Exercise

Let \( S \) be the square with vertices \( 0, 1, i \) and \( 1+i \). Draw the figure \( T \) obtained by multiplying all points of \( S \) with \( w = 1+i \).

Answer

\[
\begin{align*}
w \cdot 0 &= 0 \\
w \cdot 1 &= 1 + i \\
w(1+i) &= (1+i)^2 = 2i \\
w \cdot i &= i - 1
\end{align*}
\]

\( T \) is obtained by scaling \( S \) with a factor \( \sqrt{2} \) and by rotating \( S \) counter-clockwise over an angle of \( 45^\circ \) about \( 0 \).
Rectangular form is the same as canonical form.

Trigonometric form is like polar form but with sine and cosine, and with a non-negative angle smaller than $360^\circ$, for example:

$$z = 7 \left( \cos(225^\circ) + i \sin(225^\circ) \right).$$

MyLabsPlus uses $\text{cis}$ ("cosine plus $i$ sine") to indicate Eulers function:

$$\text{cis}(\varphi) = e^{i\varphi}.$$
In this part of the lecture we write $j$ in stead of $i$. 
### Passive components

<table>
<thead>
<tr>
<th>Component</th>
<th>Relation $v(t)$ vs. $i(t)$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Resistor</strong></td>
<td>$v(t) = Ri(t)$</td>
<td>Dissipates energy</td>
</tr>
<tr>
<td><strong>Capacitor</strong></td>
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If you know $i(t)$, then $v(t)$ can be uniquely determined.
Passive components

If you know $i(t)$, then $v(t)$ can be uniquely determined.

The component can therefore be regarded to be a system:

![Passive component diagram]

$i(t) \mapsto v(t)$

or abbreviated: $i(t) \leftrightarrow v(t)$. 
Passive components

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The component can therefore be regarded to be a system:

\[ i(t) \rightarrow S \rightarrow v(t) \]

or abbreviated: $i(t) \leftrightarrow v(t)$.

**Example:** for an inductor with inductance $L$ we have  
\[ i(t) \leftrightarrow Li'(t). \]
Definition

Let $S$ be a system. Let $x(t)$ and $y(t)$ be signals for which $S$ has the following responses:

$$x(t) \mapsto u(t) \quad \text{and} \quad y(t) \mapsto v(t).$$

Then the response of $S$ to the input $x(t) + j y(t)$ is defined as

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**Example:** for an inductor with inductance $L$ we have

$$\cos(\omega t) \mapsto -\omega L \sin(\omega t) \quad \text{and} \quad \sin(\omega t) \mapsto \omega L \cos(\omega t),$$

hence

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \mapsto -\omega L \sin(\omega t) + j\omega L \cos(\omega t) = j\omega L \left( \cos(\omega t) + j \sin(\omega t) \right) = j\omega L e^{j\omega t}.$$
Theorem

Passive components are **linear and time invariant**.

- **Linearity** means that if $x_1(t) \mapsto y_1(t)$ and $x_2(t) \mapsto y_2(t)$, then
  \[ \alpha x_1(t) + \beta x_2(t) \mapsto \alpha y_1(t) + \beta y_2(t). \]
  for all $\alpha$ and $\beta$.

- **Time invariance** means that if $x(t) \mapsto y(t)$, then
  \[ x(t - t_0) \mapsto y(t - t_0) \quad \text{for all} \ t_0. \]
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- Linear and time invariant systems are called LTI systems.
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- Linear and time invariant systems are called LTI systems.

- Passive components can be regarded as systems: the input is the current $i(t)$ through the component, and the response is the voltage $v(t)$ over the component.
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- Linear and time invariant systems are called LTI systems.
- Passive components can be regarded as systems: the input is the current \( i(t) \) through the component, and the response is the voltage \( v(t) \) over the component.
- Passive components are LTI systems.
The transfer function

**Theorem**

For all LTI systems there exists a function $Z(\omega)$ such that

\[ e^{j\omega t} \mapsto Z(\omega)e^{j\omega t} \]
The transfer function

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- The function $Z(\omega)$ is called the **transfer function**.
- The transfer function does not depend on time, but can depend on the frequency $\omega$.
- For passive components, where the input is the current $i(t)$ through the component, and the response is the voltage $v(t)$ over the component, the function $Z(\omega)$ is called the **impedance** of the component, usually denoted as $Z$. 
The transfer function

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- **Example**: for an inductor with inductance $L$ we have

\[ e^{j\omega t} \mapsto j\omega L e^{j\omega t}, \]

so the impedance is $Z = j\omega L$. 
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<thead>
<tr>
<th>Component</th>
<th>Impedance</th>
<th>Description</th>
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<tr>
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<td>$Z = R$</td>
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Example

Let \( v(t) = 5 \cos(2\pi f t) \), where the frequency is equal to \( f = 10 \text{ kHz} \). The inductance \( L \) is 50 mH. Describe the current \( i(t) \) through the inductor as a function of \( t \). What is the amplitude of \( i(t) \)?
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- The impedance of \( L \) is \( Z = j\omega L \), where \( \omega = 2\pi f \).
- Define \( \hat{v}(t) = 5e^{j\omega t} \), then

\[
\hat{i}(t) = \frac{\hat{v}(t)}{Z} = \frac{\hat{v}(t)}{j\omega L} = -\frac{5j e^{j\omega t}}{\omega L} = -\frac{5j}{2\pi f L} \left( \cos(\omega t) + j \sin(\omega t) \right)
\]

\[
= \frac{5}{2\pi f L} \left( \sin(\omega t) - j \cos(\omega t) \right).
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- The impedance of \( L \) is \( Z = j\omega L \), where \( \omega = 2\pi f \).
- Define \( \hat{v}(t) = 5 e^{j\omega t} \), then
  \[
  \hat{i}(t) = \frac{\hat{v}(t)}{Z} = \frac{\hat{v}(t)}{j\omega L} = -\frac{5j e^{j\omega t}}{\omega L} = -\frac{5j}{2\pi f L} \left( \cos(\omega t) + j \sin(\omega t) \right)
  \]
  \[
  = \frac{5}{2\pi f L} \left( \sin(\omega t) - j \cos(\omega t) \right).
  \]
- Hence \( i(t) = \text{Re} \left( \hat{i}(t) \right) = \frac{5}{2\pi f L} \sin(2\pi f t) \approx 0.001591 \sin(2\pi f t). \)
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- Define \( \hat{v}(t) = 5 e^{j\omega t} \), then
  \[
  \hat{i}(t) = \frac{\hat{v}(t)}{Z} = \frac{\hat{v}(t)}{j\omega L} = -\frac{5j e^{j\omega t}}{\omega L} = -\frac{5j}{2\pi f L} \left( \cos(\omega t) + j\sin(\omega t) \right)
  \]
  \[
  = \frac{5}{2\pi f L} \left( \sin(\omega t) - j\cos(\omega t) \right).
  \]
- Hence \( i(t) = \text{Re} \left( \hat{i}(t) \right) = \frac{5}{2\pi f L} \sin(2\pi f t) \approx 0.001591 \sin(2\pi f t) \).
- The amplitude of \( i(t) \) is 1.591 mA.
The following relations hold:

\[ v_1(t) = Z_1 i(t) \quad \text{and} \quad v_2(t) = Z_2 i(t). \]
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The voltage over clamps \( AB \) is

\[ v(t) = v_1(t) + v_2(t) = Z_1 i(t) + Z_2 i(t) = (Z_1 + Z_2)i(t). \]
The following relations hold:

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The voltage over clamps \( AB \) is

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The replacement impedance for the series composition is \( \frac{v(t)}{i(t)} \), hence

\[ Z_{\text{ser}} = Z_1 + Z_2 \]
Exercise

$C_1$ $C_2$

What is the replacement capacitance $C'$ of two capacitors in series?

The voltage over clamps $AB$ is

$$v(t) = v_1(t) + v_2(t) = Z_1 i(t) + Z_2 i(t) = (Z_1 + Z_2) i(t).$$

The replacement impedance for the series composition is

$$Z_{ser} = Z_1 + Z_2$$
Composition in series

The following relations hold:

$v_1(t) = Z_1 i(t)$ and $v_2(t) = Z_2 i(t)$.

The voltage over clamps $AB$ is $v(t) = v_1(t) + v_2(t) = Z_1 i(t) + Z_2 i(t) = (Z_1 + Z_2) i(t)$.

The replacement impedance for the series composition is $v(t) i(t)$, hence $Z_{\text{ser}} = Z_1 + Z_2$.

Exercise

What is the replacement capacitance $C'$ of two capacitors $C_1$ and $C_2$?

Answer

$$\frac{1}{j\omega C} = Z_1 + Z_2 = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} = \frac{1}{j\omega (C_1 + C_2)}'$$

hence $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ \Rightarrow $C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$. 
The total current through the circuit is

\[ i(t) = i_1(t) + i_2(t) = \frac{v(t)}{Z_1} + \frac{v(t)}{Z_2} = \left( \frac{1}{Z_1} + \frac{1}{Z_2} \right) v(t). \]
Composition in parallel

The total current through the circuit is

\[ i(t) = i_1(t) + i_2(t) = \frac{v(t)}{Z_1} + \frac{v(t)}{Z_2} = \left( \frac{1}{Z_1} + \frac{1}{Z_2} \right) v(t). \]

The reciprocal of the replacement impedance is \( \frac{i(t)}{v(t)} \), hence

\[ \frac{1}{Z_{\text{par}}} = \frac{1}{Z_1} + \frac{1}{Z_2} \]
Composition in parallel

\[ Z_1 \quad \ldots \quad Z_2 \]

**Exercise**

What is the replacement capacitance \( C' \) of two parallel capacitors?

- The reciprocal of the replacement impedance is \( \frac{v(t)}{i(t)} \), hence

\[
\frac{1}{Z_{\text{par}}} = \frac{1}{Z_1} + \frac{1}{Z_2}
\]
Composition in parallel

\[ Z_1 \cdots Z_n \]

**Exercise**

What is the replacement capacitance \( C \) of two parallel capacitors?

**Answer**

\[
\begin{align*}
  j\omega C &= \frac{1}{1/j\omega C} = \frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \\
  &= \frac{1}{1/j\omega C_1} + \frac{1}{1/j\omega C_2} = j\omega C_1 + j\omega C_2 \\
  &= j\omega (C_1 + C_2),
\end{align*}
\]

hence \( C = C_1 + C_2 \).
The for the impedance of an inductor $L$ parallel to a capacitor $C$ we have

$$\frac{1}{Z} = \frac{1}{j\omega L} + \frac{1}{1/j\omega C}$$
The for the impedance of an inductor $L$ parallel to a capacitor $C$ we have

$$\frac{1}{Z} = \frac{1}{j\omega L} + \frac{1}{1/j\omega C} = \frac{1}{j\omega L} + j\omega C$$
The impedance of an inductor $L$ parallel to a capacitor $C$ we have

$$\frac{1}{Z} = \frac{1}{j\omega L} + \frac{1}{1/j\omega C} = \frac{1}{j\omega L} + j\omega C = \frac{1 - \omega^2 LC}{j\omega L}. $$
The for the impedance of an inductor $L$ parallel to a capacitor $C$ we have

$$
\frac{1}{Z} = \frac{1}{j\omega L} + \frac{1}{1/j\omega C} = \frac{1}{j\omega L} + j\omega C = \frac{1 - \omega^2 LC}{j\omega L}.
$$

The impedance of the circuit is

$$
Z = \frac{j\omega L}{1 - \omega^2 LC}.
$$
The resonance frequency

\[ Z = \frac{j\omega L}{1 - \omega^2 LC} \]

The frequency \( \omega_{\text{res}} = \frac{1}{\sqrt{LC}} \) is called the resonance frequency.

The impedance becomes very large if \( \omega^2 \approx \frac{1}{LC} \).
The resonance frequency

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- The impedance becomes very large if \( \omega^2 \approx \frac{1}{LC} \).
- The frequency \( \omega_{\text{res}} = \frac{1}{\sqrt{LC}} \) is called the \textbf{resonance frequency}.

\[
\begin{align*}
|Z| & \quad \omega \\
\frac{1}{\sqrt{LC}} & \\
\end{align*}
\]
Answer the following questions for the circuits (1), (2) and (3).

(a) What is the replacement impedance $Z$? Write $Z$ in canonical form.

(b) What happens with $|Z|$ for high frequencies ($\omega \to \infty$)?

(c) What happens with $|Z|$ for low frequencies ($\omega \to 0$)?