This week

1. Appendix A.7: complex numbers
2. Application: impedance
A complex number is a vector in $\mathbb{R}^2$. 
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In stead of $\mathbb{R}^2$ we write $\mathbb{C}$.
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Rather than $x$- and $y$-axis, we call them the real axis and imaginary axis.
A complex number is a vector in $\mathbb{R}^2$.

In stead of $\mathbb{R}^2$ we write $\mathbb{C}$.

Rather than $x$- and $y$-axis, we call them the real axis and imaginary axis.

The complex number $i$ is defined as $(0, 1)$. 
Addition is defined termwise: if \( z = (x, y) \) and \( w = (u, v) \), then

\[
z + w = (x + y, y + v)
\]

Scalar multiplication is defined termwise: if \( z = (x, y) \) and \( \alpha \in \mathbb{R} \), then

\[
\alpha z = (\alpha x, \alpha y)
\]

Notebook: Sum.nb
Definition

Let $z = (x, y)$ and $w = (u, v)$ be two complex numbers. The product of $z$ and $w$ is defined as

$$zw = (xu - yv, xv + yu)$$
Definition

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Examples:

- $(1, 2)(4, -1) = (1 \cdot 4 - 2(-1), 1(-1) + 2 \cdot 4) = (6, 7)$. 

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\[
z w = (x u - y v, x v + y u)
\]

**Examples:**

- \((1, 2)(4, -1) = (1 \cdot 4 - 2(-1), 1(-1) + 2 \cdot 4) = (6, 7)\).
- \((2, 0)(3, -4) = (2 \cdot 3 - 0(-4), 2(-4) + 0 \cdot 3) = (2 \cdot 3, 2(-4)) = 2 (3, -4)\).
Definition

Let \( z = (x, y) \) and \( w = (u, v) \) be two complex numbers. The **product of** \( z \) **and** \( w \) **is defined as**

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Examples:

- \((1, 2)(4, -1) = (1 \cdot 4 - 2(-1), 1(-1) + 2 \cdot 4) = (6, 7)\).
- \((2, 0)(3, -4) = (2 \cdot 3 - 0(-4), 2(-4) + 0 \cdot 3) = (6, 2(-4)) = 2 (3, -4)\).
- \(i^2 = i \cdot i = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)\).

**Notebook: Product.nb**
Definition

Let \( z = (x, y) \) and \( w = (u, v) \) be two complex numbers. The product of \( z \) and \( w \) is defined as

\[
z w = (x u - y v, x v + y u)
\]

Exercise

Let \( z = (2, 1) \) and \( w = (3, 1) \), calculate \( zw \).

- \((2, 0)(3, -4) = (2 \cdot 3 - 0 \cdot (-4), 2(-4) + 0 \cdot 3) = (6, -8) = 2(3, -4)\).
- \(i^2 = i \cdot i = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)\).

Notebook: Product.nb
Definition
Let $z = (x, y)$ and $w = (u, v)$ be two complex numbers. The **product of $z$ and $w$** is defined as

$$zw = (x u - y v, x v + y u)$$

**Exercise**
Let $z = (2, 1)$ and $w = (3, 1)$, calculate $zw$.

**Answer**
$$zw = (5, 5).$$
Convention

Every real number $x$ is identified with the complex number $(x, 0)$.
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Examples: $0 = (0, 0)$, $1 = (1, 0)$, $-1 = (-1, 0)$. 
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Every real number $x$ is identified with the complex number $(x, 0)$.

- Examples: $0 = (0, 0)$, $1 = (1, 0)$, $-1 = (-1, 0)$.

- The complex numbers on the real axis behave just like the real numbers in $\mathbb{R}$:
  - $x + y \rightarrow (x, 0) + (y, 0) = (x + y, 0 + 0) = (x + y, 0)$.
  - $x - y \rightarrow (x, 0) - (y, 0) = (x - y, 0 - 0) = (x - y, 0)$.
  - $xy \rightarrow (x, 0)(y, 0) = (xy - 0 \cdot 0, x \cdot 0 + 0 \cdot y) = (xy, 0)$. 
By identifying $x \in \mathbb{R}$ with the complex number $(x, 0)$, we regard the points on the real axis as the real number line.
Real numbers are complex numbers

- By identifying $x \in \mathbb{R}$ with the complex number $(x, 0)$, we regard the points on the real axis as the real number line.

- $i^2 = -1$
Real numbers are complex numbers

- By identifying \( x \in \mathbb{R} \) with the complex number \((x, 0)\), we regard the points on the real axis as the real number line.

- \( i^2 = -1 \)

- The complex numbers are an expansion of the real numbers:
Let \( z, w \) and \( u \) be complex numbers. Define \( z - w \) and \( -z \) in the usual way, then

1. \( z + w = w + z \)
2. \( z + w + u = z + (w + u) = (z + w) + u \)
3. \( z + 0 = z \)
4. \( -z = (\text{--}1)z \)
5. \( z - w = z + (-w) \)
6. \( z - z = 0 \)
7. \( zw = wz \)
8. \( z \cdot 1 = z \)
9. \( z \cdot 0 = 0 \)
10. \( z w u = z (w u) = (z w) u \)
11. \( z(w + u) = zw + zu \)
12. \( z(w - u) = zw - zu \)
The canonical form

**Theorem**

Let \( z = (x, y) \) be a complex number, with \( x \) and \( y \) real. Then

\[
z = x + iy.
\]
The canonical form

**Theorem**

Let $z = (x, y)$ be a complex number, with $x$ and $y$ real. Then

$$z = x + iy.$$  

**Proof:**

$$x + iy = (x, 0) + (0, 1)(y, 0)$$
The canonical form

**Theorem**

Let $z = (x, y)$ be a complex number, with $x$ and $y$ real. Then

$$z = x + i y.$$ 

**Proof:**

$$x + i y = (x, 0) + (0, 1)(y, 0)$$

$$= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y)$$
Theorem

Let \( z = (x, y) \) be a complex number, with \( x \) and \( y \) real. Then

\[
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Proof:

\[
x + i y = (x, 0) + (0, 1)(y, 0)
\]

\[
= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y)
\]

\[
= (x, 0) + (0, y) = (x, y) = z.
\]
The canonical form

Theorem

Let \( z = (x, y) \) be a complex number, with \( x \) and \( y \) real. Then

\[
z = x + i \, y.
\]

Proof:

\[
x + i \, y = (x, 0) + (0, 1)(y, 0)
\]
\[
= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y)
\]
\[
= (x, 0) + (0, y) = (x, y) = z.
\]

Definition

The form \( x + i \, y \) is called the \textbf{canonical form} of \( z \).
The canonical form

**Theorem**

Let \( z = (x, y) \) be a complex number, with \( x \) and \( y \) real. Then

\[
    z = x + i y.
\]

**Proof:**

\[
    x + i y = (x, 0) + (0, 1)(y, 0) \\
    = (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y) \\
    = (x, 0) + (0, y) = (x, y) = z.
\]

**Definition**

The form \( x + i y \) is called the **canonical form** of \( z \).

- Henceforth we will always write complex numbers in canonical form.
Let $z = x + i y$ and $w = u + i v$ be two complex numbers, with $x$, $y$, $u$ and $v$ real. Then

\[
z + w = (x + i y) + (u + i v)\\
= x + u + i y + i v\\
= (x + u) + i(y + v).
\]
Let \( z = x + i y \) and \( w = u + i v \) be two complex numbers, with \( x, y, u \) and \( v \) real. Then

\[
z + w = (x + i y) + (u + i v)
\]

\[
= x + u + i y + i v
\]

\[
= (x + u) + i(y + v).
\]

For the product of \( z \) and \( w \) we have

\[
z w = (x + i y)(u + i v)
\]
Let $z = x + i y$ and $w = u + i v$ be two complex numbers, with $x$, $y$, $u$ and $v$ real. Then

$$z + w = (x + i y) + (u + i v)$$

$$= x + u + i y + i v$$

$$= (x + u) + i(y + v).$$

For the product of $z$ and $w$ we have

$$z w = (x + i y)(u + i v)$$

$$= x u + (i y)(i v) + x(i v) + (i y)u$$
Let \( z = x + i\, y \) and \( w = u + i\, v \) be two complex numbers, with \( x, y, u \) and \( v \) real. Then

\[
z + w = (x + i\, y) + (u + i\, v) \\
= x + u + i\, y + i\, v \\
= (x + u) + i\,(y + v).
\]

For the product of \( z \) and \( w \) we have

\[
z\, w = (x + i\, y)(u + i\, v) \\
= x\, u + (i\, y)(i\, v) + x\,(i\, v) + (i\, y)\, u \\
= x\, u + i^2\, y\, v + i\, x\, v + i\, y\, u
\]
Let $z = x + i\, y$ and $w = u + i\, v$ be two complex numbers, with $x$, $y$, $u$ and $v$ real. Then

$$z + w = (x + i\, y) + (u + i\, v)$$
$$= x + u + i\, y + i\, v$$
$$= (x + u) + i( y + v).$$

For the product of $z$ and $w$ we have

$$z\, w = (x + i\, y)(u + i\, v)$$
$$= x\, u + (i\, y)(i\, v) + x(i\, v) + (i\, y)u$$
$$= x\, u + i^2 y\, v + i\, x\, v + i\, y\, u$$
$$= (x\, u - y\, v) + i(x\, v + y\, u).$$
Let \( z = x + iy \) and \( w = u + iv \) be two complex numbers, with \( x, y, u \) and \( v \) real. Then

\[
z + w = (x + iy) + (u + iv)
\]

**Exercise**

Calculate \((3 - 2i)^2\).

\[
z w = (x + iy)(u + iv)
\]

\[
= xu + (iy)(iv) + xi v + iy u
\]

\[
= xu + i^2yv + ixv + iyu
\]

\[
= (xu - yv) + i(xv + yu).
\]
Let \( z = x + iy \) and \( w = u + iv \) be two complex numbers, with \( x, y, u \) and \( v \) real. Then

\[
z + w = (x + iy) + (u + iv)
\]

**Exercise**

Calculate \((3 - 2i)^2\).

**Answer**

\[(3 - 2i)^2 = 5 - 12i.\]
Assignment: IMM2 - Tutorial 6.1
Real and Imaginary part

**Definition**

Let $z = x + i\ y$ be a complex number with $x$ and $y$ real. Then $x$ is the \textbf{real part of} $z$ and $y$ is the \textbf{imaginary part of} $z$. We denote

$$x = \text{Re} z \quad \text{and} \quad y = \text{Im} z.$$
2.2 Definition

Let $z = x + i \, y$ be a complex number with $x$ and $y$ real. Then the conjugate of $z$ is the complex number $\bar{z}$ defined by

$$\bar{z} = x - i \, y.$$
**Definition**

Let \( z = x + iy \) be a complex number with \( x \) and \( y \) real. Then the **absolute value of** \( z \) is the distance of \( z \) to 0:

\[
|z| = \sqrt{x^2 + y^2}.
\]

- The definition is based on the Pythagorean theorem.
- The absolute value is sometimes called **modulus** or **norm**.
Let $z$ and $w$ be complex numbers, then

1. $z + w = \bar{z} + \bar{w}$

2. $z - w = \bar{z} - \bar{w}$

3. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

4. $|z|^2 = z \bar{z}$

5. $|z \cdot w| = |z| \cdot |w|$

6. $|z + w| \leq |z| + |w|$
Let $z$ and $w$ be complex numbers, then

1. $\overline{z + w} = \overline{z} + \overline{w}$

2. $\overline{z - w} = \overline{z} - \overline{w}$

3. $\overline{zw} = \overline{z} \overline{w}$

4. $|z|^2 = z \overline{z}$

5. $|zw| = |z| |w|$

6. $|z + w| \leq |z| + |w|$

Property 6 is called the **triangular inequality**.
Let $z$ and $w$ be complex numbers, then

1. $\overline{z + w} = \overline{z} + \overline{w}$
2. $\overline{z - w} = \overline{z} - \overline{w}$
3. $zw = z\overline{w}$
4. $|z|^2 = z\overline{z}$
5. $|zw| = |z||w|

**Exercise**

Let $z = 2 + i$, calculate $|z|$, $|z|^2$ and $|z^2|$. 
Properties of conjugation and absolute value

Let $z$ and $w$ be complex numbers, then

1. $z + w = \overline{z} + \overline{w}$
2. $z - w = \overline{z} - \overline{w}$
3. $zw = \overline{z} \overline{w}$
4. $|z|^2 = z \overline{z}$
5. $|zw| = |z| |w|

Exercise

Let $z = 2 + i$, calculate $|z|$, $|z|^2$ and $|z^2|$.

Answer

$|z| = \sqrt{5}$

$|z|^2 = |z^2| = 5$  ($|z|^2 = |z^2|$ follows from 5.)
**Theorem**

*For every complex number* \( z \) *the following holds:*

1. \( \text{Re} \, z = \frac{\overline{z} + z}{2} \)
2. \( \text{Im} \, z = \frac{\overline{z} - z}{2i} \)

*Write* \( z = x + iy \), then

\[
\overline{z} + z = (x + iy) + (x - iy) = 2x = 2 \text{ Re} \, z,
\]

\[
\overline{z} - z = (x + iy) - (x - iy) = 2iy = 2 \text{ Im} \, z.
\]
The real and imaginary part

**Theorem**

For every complex number $z$ the following holds:

1. $\text{Re } z = \frac{\bar{z} + z}{2}$
2. $\text{Im } z = \frac{\bar{z} - z}{2i}$

- Write $z = x + iy$, then

$$z + \bar{z} = (x + iy) + (x - iy)$$
The real and imaginary part

**Theorem**

For every complex number $z$ the following holds:

1. $\text{Re } z = \frac{\bar{z} + z}{2}$
2. $\text{Im } z = \frac{\bar{z} - z}{2}i$

- Write $z = x + iy$, then

\[
    z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\text{ Re } z,
\]
The real and imaginary part

Theorem

For every complex number $z$ the following holds:

(1) $\text{Re } z = \frac{\overline{z} + z}{2}$

(2) $\text{Im } z = \frac{\overline{z} - z}{2i}$

Write $z = x + iy$, then

\[ z + \overline{z} = (x + iy) + (x - iy) = 2x = 2 \text{ Re } z, \]

and

\[ z - \overline{z} = (x + iy) - (x - iy) \]
The real and imaginary part

Theorem

For every complex number \( z \) the following holds:

1. \( \text{Re } z = \frac{\bar{z} + z}{2} \)
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z + \bar{z} = (x + iy) + (x - iy) = 2x = 2 \text{ Re } z,
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and

\[
z - \bar{z} = (x + iy) - (x - iy) = 2iy,
\]
The real and imaginary part

**Theorem**

For every complex number $z$ the following holds:

1. $\text{Re } z = \frac{z + \bar{z}}{2}$
2. $\text{Im } z = \frac{z - \bar{z}}{2i}$

Write $z = x + iy$, then

\[ z + \bar{z} = (x + iy) + (x - iy) = 2x = 2 \text{ Re } z, \]

and

\[ z - \bar{z} = (x + iy) - (x - iy) = 2iy, \]

\[ -\frac{1}{2}i(z - \bar{z}) = y = \text{ Im } z. \]
Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$. 

The number $w$ is called the reciproke of $z$ and is denoted as $1/z$. The reciproke of $z$ is sometimes denoted as $z^{-1}$. If $z = x + iy$ with $x$ and $y$ real, then $1/z = 1/|z|^2 z = 1/(x^2 + y^2)(x - iy) = x/(x^2 + y^2) - y/(x^2 + y^2)i$. 

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Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$.

- Assume that $zw = 1$, then

$$
\bar{z} \cdot w = \bar{z}
$$
Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$.

- Assume that $zw = 1$, then

\[
\bar{z} z w = \bar{z} \implies |z|^2 w = \bar{z}
\]
Problem

For arbitrary $z \neq 0$, find a complex number $w$ such that $zw = 1$.

- Assume that $zw = 1$, then

\[
\bar{z}zw = \bar{z} \quad \Rightarrow \quad |z|^2 w = \bar{z} \quad \Rightarrow \quad w = \frac{1}{|z|^2} \bar{z}
\]
The reciproke

Problem

For arbitrary \( z \neq 0 \), find a complex number \( w \) such that \( zw = 1 \).

- Assume that \( zw = 1 \), then

\[
\overline{z}zw = \overline{z} \quad \Rightarrow \quad |z|^2 w = \overline{z} \quad \Rightarrow \quad \frac{1}{z} = w = \frac{1}{|z|^2} \overline{z}
\]

- The number \( w \) is called the \textbf{reciproke of} \( z \) and is denoted as \( \frac{1}{z} \).
The reciprocal

Problem

For arbitrary \( z \neq 0 \), find a complex number \( w \) such that \( zw = 1 \).

- Assume that \( zw = 1 \), then
  \[
  \overline{z}zw = \overline{z} \quad \Rightarrow \quad |z|^2 w = \overline{z} \quad \Rightarrow \quad \frac{1}{z} = w = \frac{1}{|z|^2} \overline{z}
  \]

- The number \( w \) is called the **reciproke** of \( z \) and is denoted as \( \frac{1}{z} \).

- The reciproke of \( z \) is sometimes denoted as \( z^{-1} \).
The reciproke

Problem

For arbitrary \( z \neq 0 \), find a complex number \( w \) such that \( z \, w = 1 \).

- Assume that \( z \bar{w} = 1 \), then

\[
\begin{align*}
\bar{z} \, z \, w &= \bar{z} \quad \Rightarrow \quad |z|^2 \, w = \bar{z} \quad \Rightarrow \\
\frac{1}{z} &= w = \frac{1}{|z|^2} \, \bar{z}
\end{align*}
\]

- The number \( w \) is called the **reciproke of** \( z \) and is denoted as \( \frac{1}{z} \).

- The reciproke of \( z \) is sometimes denoted as \( z^{-1} \).

- If \( z = x + iy \) with \( x \) and \( y \) real, then

\[
\frac{1}{z} = \frac{1}{|z|^2} \, \bar{z} = \frac{1}{x^2 + y^2} (x - iy) = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} \, i.
\]
**Definition**

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the **quotient of $u$ and $z$** is defined as the product of $u$ and the reciprocal of $z$:  

\[
\frac{u}{z} = u \cdot \frac{1}{z}.
\]
**Definition**

Let \( z \) and \( w \) be complex numbers. If \( z \neq 0 \) then the **quotient of** \( u \) and \( z \) is defined as the product of \( u \) and the reciprocal of \( z \):

\[
\frac{u}{z} = u \cdot \frac{1}{z}.
\]

**Notebook**: Quotient.nb

- Equivalently we can write \( \frac{u}{z} = \frac{1}{|z|^2} u \bar{z} \).

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**Example:**

\[
\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{5} = 1 - i.
\]
**Definition**

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$\frac{u}{z} = u \cdot \frac{1}{z}.$$

- Equivalently we can write $\frac{u}{z} = \frac{1}{|z|^2} \cdot u \overline{z}$.
- Practical approach: multiply numerator and denominator with $\overline{z}$:

$$\frac{u}{z} = \frac{u \overline{z}}{z \overline{z}},$$

and elaborate $u \overline{z}$. 

**Example:**

$$\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{5} = 1 - i.$$
Definition

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$
\frac{u}{z} = u \cdot \frac{1}{z}.
$$

- Equivalently we can write $\frac{u}{z} = \frac{1}{|z|^2} u \overline{z}$.
- Practical approach: multiply numerator and denominator with $\overline{z}$:
  $$
  \frac{u}{z} = \frac{u \overline{z}}{z \overline{z}},
  $$
  and elaborate $u \overline{z}$.
- Example:
  $$
  \frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)}
  $$
Definition

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciprocal of $z$:

$$\frac{u}{z} = u \cdot \frac{1}{z}.$$ 

Equivalently we can write

$$\frac{u}{z} = \frac{1}{|z|^2} u \bar{z}.$$ 

Practical approach: multiply numerator and denominator with $\bar{z}$:

$$\frac{u}{z} = \frac{u \bar{z}}{z \bar{z}},$$

and elaborate $u \bar{z}$.

Example:

$$\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{1^2 + 2^2}$$
Definition

Let \( z \) and \( w \) be complex numbers. If \( z \neq 0 \) then the quotient of \( u \) and \( z \) is defined as the product of \( u \) and the reciprocal of \( z \):

\[
\frac{u}{z} = \frac{1}{\bar{z}}.
\]

- Equivalently we can write \( \frac{u}{z} = \frac{1}{|z|^2} \frac{u}{\bar{z}} \).
- Practical approach: multiply numerator and denominator with \( \bar{z} \):

\[
\frac{u}{z} = \frac{u \bar{z}}{z \bar{z}},
\]

and elaborate \( u \bar{z} \).
- Example:

\[
\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{1^2 + 2^2} = \frac{5 - 5i}{5} = 1 - i.
\]
**Definition**

Let $z$ and $w$ be complex numbers. If $z \neq 0$ then the quotient of $u$ and $z$ is defined as the product of $u$ and the reciproke of $z$:

$$u / z = u \frac{1}{z}.$$

Equivalently we can write

$$u / z = \frac{1}{|z|^2} u z.$$

**Exercise**

Write $\frac{3 + i}{3 - 4i}$ in canonical form.

**Practical approach:** multiply numerator and denominator with $\bar{z}$:

$$u / z = \frac{u \bar{z}}{z \bar{z}},$$

and elaborate $u \bar{z}$.

**Example:**

$$\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{1^2 + 2^2} = \frac{5 - 5i}{5} = 1 - i.$$
**Definition**

Let $z$ and $w$ be complex numbers. If $w \neq 0$, then the quotient of $z$ and $w$ is defined as the product of $z$ and the reciprocal of $w$:

$$z \div w = \frac{z}{w}$$

Equivalently, we can write

$$z \div w = \frac{|w|^2}{z \cdot \overline{w}}.$$

**Exercise**

Write $\frac{3 + i}{3 - 4i}$ in canonical form.

**Answer**

$$\frac{3 + i}{3 - 4i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{5} = 1 - i.$$
Let $u \neq 0$, $v$, $z \neq 0$ and $w$ be complex numbers.

1. $\frac{w}{1} = w$

2. $\frac{w}{z} \frac{v}{u} = \frac{w}{z} \frac{v}{u}$

3. $\frac{1}{w/z} = \frac{z}{w}$ (for $w \neq 0$)

4. $\frac{w}{z} = \frac{\bar{w}}{\bar{z}}$

5. $\left| \frac{w}{z} \right| = \left| \frac{w}{z} \right|$

For all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ the following holds:

1. $z^m z^n = z^{m+n}$

2. $(z^m)^n = z^{mn}$

3. $\frac{1}{z^m} = z^{-m}$

4. $z^n w^n = (zw)^n$

5. $\left( \frac{w}{z} \right)^n = \frac{w^n}{z^n}$
Assignment: IMM2 - Tutorial 6.2
The argument of a complex number $z \neq 0$ is the angle that the line through 0 and $z$ makes with the positive real axis. The argument of $z$ is denoted as $\text{arg}(z)$.

- The argument of 0 is not defined.
**Definition**

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- The argument of 0 is not defined.
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- If the direction is clockwise, the argument is negative.
Definition

The **argument** of a complex number $z \neq 0$ is the angle that the line through 0 and $z$ makes with the positive real axis.

The argument of $z$ is denoted as $\arg(z)$.

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- The argument is expressed in radians.
- The argument is measured from the positive real axis.
- If the direction is counter-clockwise, the argument is *positive*.
- If the direction is clockwise, the argument is *negative*.
- The argument is determined up to a multiple of $2\pi$. 

**Exercise**

Find $\arg(z)$ for $z = 1$, $z = i$, $z = -1$ and $z = -i$.

**Answer**

$\arg(1) = 0$

$\arg(i) = \frac{\pi}{2}$

$\arg(-1) = \pi$

$\arg(-i) = -\frac{\pi}{2}$ or $\frac{3\pi}{2}$
The argument of a complex number $z \neq 0$ is the angle that the line through 0 and $z$ makes with the positive real axis. The argument of $z$ is denoted as $\arg(z)$.

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**Exercise**

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The Euler function

**Definition**

*The Euler function is the function that assigns to every real number* \( \varphi \) *the complex number*

\[
e^{i \varphi} = \cos \varphi + i \sin \varphi.
\]
The Euler function

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e^{i\varphi} = \cos \varphi + i \sin \varphi.
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- The number \( e^{i\varphi} \) lies on the unit circle: \( |e^{i\varphi}| = 1 \).
The Euler function

**Definition**

The Euler function is the function that assigns to every real number $\varphi$ the complex number

$$e^{i\varphi} = \cos \varphi + i \sin \varphi.$$

- The number $e^{i\varphi}$ lies on the unit circle: $|e^{i\varphi}| = 1$.
- The real part of $e^{i\varphi}$ is $\cos \varphi$, the imaginary part of $e^{i\varphi}$ is $\sin \varphi$. 
The Euler function is the function that assigns to every real number \( \varphi \) the complex number

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- The number \( e^{i\varphi} \) lies on the unit circle: \( |e^{i\varphi}| = 1 \).
- The real part of \( e^{i\varphi} \) is \( \cos \varphi \), the imaginary part of \( e^{i\varphi} \) is \( \sin \varphi \).
- The complex number \( e^{i\varphi} \) is the number on the unit circle with argument \( \varphi \).
The Euler function

**Theorem**

For every real number $\varphi$ and $\psi$ we have

$$e^{i(\varphi + \psi)} = e^{i\varphi} e^{i\psi}$$
The Euler function

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For every real number $\varphi$ and $\psi$ we have

$$e^{i(\varphi+\psi)} = e^{i\varphi} e^{i\psi}$$

- Use trigonometry formulas to derive

  $$e^{i(\varphi+\psi)} = \cos(\varphi + \psi) + i \sin(\varphi + \psi)$$
  
  $$= \cos \varphi \cos \psi - \sin \varphi \sin \psi + i(\sin \varphi \cos \psi + \cos \varphi \sin \psi).$$
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For every real number $\varphi$ and $\psi$ we have

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- Expand the right-hand side:

  $$e^{i\varphi} e^{i\psi} = (\cos \varphi + i \sin \varphi)(\cos \psi + i \sin \psi)$$
  $$= \cos \varphi \cos \psi + i^2 \sin \varphi \sin \psi + i \sin \varphi \cos \psi + i \cos \varphi \sin \psi$$
The Euler function

**Theorem**

For every real number \( \varphi \) and \( \psi \) we have

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e^{i(\varphi + \psi)} = e^{i\varphi} e^{i\psi}
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\[
= \cos \varphi \cos \psi - \sin \varphi \sin \psi + i(\sin \varphi \cos \psi + \cos \varphi \sin \psi)
\]

\[
= e^{i(\varphi + \psi)}.
\]
The Euler function Cheat Sheet

\[ e^{i\varphi} = \cos \varphi + i \sin \varphi \]

\[ e^{i0} = 1 \]

\[ |e^{i\varphi}| = 1 \]

\[ e^{i(\varphi + \psi)} = e^{i\varphi} e^{i\psi} \]

\[ (e^{i\varphi})^n = e^{in\varphi} \text{ for all } n \in \mathbb{Z} \]

\[ e^{i\varphi} = e^{-i\varphi} = \frac{1}{e^{i\varphi}} \]
**Theorem**

*Every complex number* $z \neq 0$ *can be written as the product of a positive real number and an Euler function value. In particular, if* $r = |z|$ *and* $\varphi = \arg z$, *then*

$$z = r e^{i \varphi}$$
Theorem

Every complex number \( z \neq 0 \) can be written as the product of a positive real number and an Euler function value. In particular, if \( r = |z| \) and \( \varphi = \arg z \), then

\[
z = r e^{i\varphi}
\]

- Write \( z = x + i y \) with \( x \) and \( y \) real, then

\[
\cos \varphi = \frac{x}{r} \quad \text{and} \quad \sin \varphi = \frac{y}{r}.
\]
Theorem

Every complex number $z \neq 0$ can be written as the product of a positive real number and an Euler function value. In particular, if $r = |z|$ and $\varphi = \arg z$, then

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Write $z = x + i y$ with $x$ and $y$ real, then

$$\cos \varphi = \frac{x}{r} \quad \text{and} \quad \sin \varphi = \frac{y}{r}.$$
Theorem

Let \( z \) and \( w \) be two complex numbers written in polar coordinates:

\[
z = r \, e^{i\varphi} \quad \text{and} \quad w = s \, e^{i\psi},
\]

then

\[
z w = r s e^{i(\varphi + \psi)} \quad \text{and (if } w \neq 0) \quad \frac{z}{w} = \frac{r}{s} e^{i(\varphi - \psi)}.
\]
Theorem

Let $z$ and $w$ be two complex numbers written in polar coordinates:

$$z = r e^{i\varphi} \quad \text{and} \quad w = s e^{i\psi},$$

then

$$z \, w = r s e^{i(\varphi+\psi)} \quad \text{and (if } w \neq 0) \quad \frac{z}{w} = \frac{r}{s} e^{i(\varphi-\psi)}.$$

In other words:

- the absolute value of $z \, w$ is the **product** of $|z|$ and $|w|$,
- the argument of $z \, w$ is the **sum** of $\arg z$ and $\arg w$,
- the absolute value of $z/w$ is the **quotient** of $|z|$ and $|w|$,
- the argument of $z/w$ is the **difference** of $\arg z$ and $\arg w$. 
Corollary

Let \( w = r e^{i\varphi} \). Then multiplication of an arbitrary complex number \( z \) with \( w \) can be constructed geometrically by scaling \( z \) with scale factor \( r \), and by rotating \( z \) over an angle \( \varphi \) about 0.
**Corollary**

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- **Example:** let \( w = \sqrt{3} + i = 2 e^{i\pi/6} \), then \( zw \) is obtained by scaling \( z \) with factor 2, and by rotating \( z \) over an angle of 30°.
Corollary

Let \( w = r e^{i\varphi} \). Then multiplication of an arbitrary complex number \( z \) with \( w \) can be constructed geometrically by scaling \( z \) with scale factor \( r \), and by rotating \( z \) over an angle \( \varphi \) about 0.

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**Exercise**

Let $S$ be the square with vertices 0, 1, $i$ and $1 + i$. Draw the figure $T$ obtained by multiplying all points of $S$ with $w = 1 + i$.

**Example**: let $w = \sqrt{3} + i = 2e^{i\pi/6}$, then $zw$ is obtained by scaling $z$ with factor 2, and by rotating $z$ over an angle of 30°.
Corollary

Let $w = re^{i\phi}$. Then multiplication of an arbitrary complex number $z$ with $w$ can be constructed geometrically by scaling $z$ with scale factor $r$, and by rotating $z$ over an angle $\phi$ about 0.

Example: let $w = \sqrt{3} + i = 2e^{i\pi/6}$, then $zw$ is obtained by scaling $z$ with factor 2, and by rotating $z$ over an angle of $30^\circ$.

Exercise

Let $S$ be the square with vertices 0, 1, $i$ and $1 + i$. Draw the figure $T$ obtained by multiplying all points of $S$ with $w = 1 + i$.

Answer

$$w \cdot 0 = 0$$
$$w \cdot 1 = 1 + i$$
$$w(1 + i) = (1 + i)^2 = 2i$$
$$w \cdot i = i - 1$$

$T$ is obtained by scaling $S$ with a factor $\sqrt{2}$ and by rotating $S$ counter-clockwise over an angle of $45^\circ$ about 0.
■ **Rectangular form** is the same as canonical form.

■ **Trigonometric form** is like polar form but with sine and cosine, and with a non-negative angle smaller than $360^\circ$, for example:

$$z = 7\left( \cos(225^\circ) + i \sin(225^\circ) \right).$$

■ MyLabsPlus uses $\text{cis}$ (“cosine plus $i$ sine”) to indicate Eulers function:

$$\text{cis}(\varphi) = e^{i\varphi}.$$
In this part of the lecture we write $j$ in stead of $i$. 
<table>
<thead>
<tr>
<th>Component</th>
<th>Relation ( v(t) ) vs. ( i(t) )</th>
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<td>Resistor</td>
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Passive components

- If you know $i(t)$, then $v(t)$ can be uniquely determined.
Passive components

If you know $i(t)$, then $v(t)$ can be uniquely determined.

The component can therefore be regarded to be a system:

\[ i(t) \rightarrow S \rightarrow v(t) \]

or abbreviated: $i(t) \mapsto v(t)$. 
If you know $i(t)$, then $v(t)$ can be uniquely determined.

The component can therefore be regarded to be a system:

$$i(t) \rightarrow S \rightarrow v(t)$$

or abbreviated: $i(t) \mapsto v(t)$.

**Example:** for an inductor with inductance $L$ we have

$$i(t) \mapsto Li'(t).$$
**Definition**

Let $S$ be a system. Let $x(t)$ and $y(t)$ be signals for which $S$ has the following responses:

$$x(t) \mapsto u(t) \quad \text{and} \quad y(t) \mapsto v(t).$$

Then the response of $S$ to the input $x(t) + j y(t)$ is defined as

$$u(t) + j v(t).$$

Example: for an inductor with inductance $L$ we have

$$\cos(\omega t) \mapsto -\omega L \sin(\omega t) \quad \text{and} \quad \sin(\omega t) \mapsto \omega L \cos(\omega t),$$

hence

$$e^{j \omega t} = \cos(\omega t) + j \sin(\omega t) \mapsto -\omega L \sin(\omega t) + j \omega L \cos(\omega t) = j \omega L (\cos(\omega t) + j \sin(\omega t)) = j \omega L e^{j \omega t}.$$
**Definition**

Let $S$ be a system. Let $x(t)$ and $y(t)$ be signals for which $S$ has the following responses:

\[ x(t) \mapsto u(t) \quad \text{and} \quad y(t) \mapsto v(t). \]

Then the response of $S$ to the input $x(t) + jy(t)$ is defined as

\[ u(t) + jv(t). \]

**Example:** for an inductor with inductance $L$ we have

\[ \cos(\omega t) \mapsto -\omega L \sin(\omega t) \quad \text{and} \quad \sin(\omega t) \mapsto \omega L \cos(\omega t), \]

hence

\[ e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \mapsto -\omega L \sin(\omega t) + j\omega L \cos(\omega t) \]

\[ = j\omega L \left( \cos(\omega t) + j\sin(\omega t) \right) \]

\[ = j\omega L e^{j\omega t}. \]
Passive components are linear and time invariant.

- **Linearity** means that if $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$, then
  \[
  \alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t).
  \]
  for all $\alpha$ and $\beta$.

- **Time invariance** means that if $x(t) \rightarrow y(t)$, then
  \[
  x(t - t_0) \rightarrow y(t - t_0) \quad \text{for all } t_0.
  \]
Theorem

Passive components are **linear** and **time invariant**.

- **Linearity** means that if \( x_1(t) \mapsto y_1(t) \) and \( x_2(t) \mapsto y_2(t) \), then
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Linear and time invariant systems are called LTI systems.
Theorem

Passive components are **linear and time invariant**.

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Linear and time invariant systems are called LTI systems.

Passive components can be regarded as systems: the input is the current $i(t)$ through the component, and the response is the voltage $v(t)$ over the component.
Theorem

Passive components are linear and time invariant.

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Linear and time invariant systems are called LTI systems.

Passive components can be regarded as systems: the input is the current $i(t)$ through the component, and the response is the voltage $v(t)$ over the component.

Passive components are LTI systems.
Theorem

For all LTI systems there exists a function $Z(\omega)$ such that

\[ e^{j\omega t} \mapsto Z(\omega) e^{j\omega t} \]
Theorem

For all LTI systems there exists a function \( Z(\omega) \) such that

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e^{j\omega t} \mapsto Z(\omega) e^{j\omega t}
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- The function \( Z(\omega) \) is called the **transfer function**.
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For all LTI systems there exists a function $Z(\omega)$ such that

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- The transfer function does not depend on time, but can depend on the frequency $\omega$. 
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For all LTI systems there exists a function $Z(\omega)$ such that

$$e^{j\omega t} \mapsto Z(\omega)e^{j\omega t}$$

- The function $Z(\omega)$ is called the **transfer function**.
- The transfer function does not depend on time, but can depend on the frequency $\omega$.
- For passive components, where the input is the current $i(t)$ through the component, and the response is the voltage $v(t)$ over the component, the function $Z(\omega)$ is called the **impedance** of the component, usually denoted as $Z$. 
The transfer function

**Theorem**

For all LTI systems there exists a function $Z(\omega)$ such that

\[ e^{j\omega t} \mapsto Z(\omega) e^{j\omega t} \]

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- For passive components, where the input is the current $i(t)$ through the component, and the response is the voltage $v(t)$ over the component, the function $Z(\omega)$ is called the **impedance** of the component, usually denoted as $Z$.
- **Example**: for an inductor with inductance $L$ we have

  \[ e^{j\omega t} \mapsto j\omega L e^{j\omega t}, \]

  so the impedance is $Z = j\omega L$. 
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Example

Let \( v(t) = 5 \cos(2\pi f t) \), where the frequency is equal to \( f = 10 \text{ kHz} \). The inductance \( L \) is 50 mH. Describe the current \( i(t) \) through the inductor as a function of \( t \). What is the amplitude of \( i(t) \)?
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- The impedance of $L$ is $Z = j\omega L$, where $\omega = 2\pi f$. 

\[ L \quad \text{\(i(t)\)} \]
\[ v(t) \]
\[ L = 10 \text{ mH} \]
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Let \( v(t) = 5 \cos(2\pi f t) \), where the frequency is equal to \( f = 10 \text{ kHz} \). The inductance \( L \) is \( 50 \text{ mH} \). Describe the current \( i(t) \) through the inductor as a function of \( t \). What is the amplitude of \( i(t) \)?

- The impedance of \( L \) is \( Z = j\omega L \), where \( \omega = 2\pi f \).
- Define \( \hat{v}(t) = 5e^{j\omega t} \), then

\[
\hat{i}(t) = \frac{\hat{v}(t)}{Z} = \frac{\hat{v}(t)}{j\omega L} = -\frac{5j e^{j\omega t}}{\omega L} = -\frac{5j}{2\pi f L} \left( \cos(\omega t) + j \sin(\omega t) \right)
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  \]
- Hence \( i(t) = \text{Re} \left( \hat{i}(t) \right) = \frac{5}{2\pi f L} \sin(2\pi f t) \approx 0.001591 \sin(2\pi f t) \).
Impedance of an inductor

Example

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- The impedance of \( L \) is \( Z = j\omega L \), where \( \omega = 2\pi f \).
- Define \( \hat{v}(t) = 5e^{j\omega t} \), then
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- Hence \( i(t) = \text{Re} \left( \hat{i}(t) \right) = \frac{5}{2\pi f L} \sin(2\pi f t) \approx 0.001591 \sin(2\pi f t) \).
- The amplitude of \( i(t) \) is 1.591 mA.
Composition in series

The following relations hold:

\[ v_1(t) = Z_1 i(t) \quad \text{and} \quad v_2(t) = Z_2 i(t). \]
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The voltage over clamps \( AB \) is
\[ v(t) = v_1(t) + v_2(t) = Z_1 i(t) + Z_2 i(t) = (Z_1 + Z_2)i(t). \]
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The replacement impedance for the series composition is \( \frac{v(t)}{i(t)} \), hence

\[ Z_{\text{ser}} = Z_1 + Z_2 \]
Exercise

What is the replacement capacitance $C'$ of two capacitors in series?

The voltage over clamps $AB$ is

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The replacement impedance for the series composition is $\frac{v(t)}{i(t)}$, hence

$$Z_{\text{ser}} = Z_1 + Z_2.$$
Exercise

What is the replacement capacitance $C'$ of two capacitors

Answer

$$\frac{1}{j\omega C} = Z_1 + Z_2 = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} = \frac{1}{j\omega (C_1 + C_2)}.$$ 

Hence

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \implies C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}.$$
The total current through the circuit is

\[ i(t) = i_1(t) + i_2(t) = \frac{v(t)}{Z_1} + \frac{v(t)}{Z_2} = \left( \frac{1}{Z_1} + \frac{1}{Z_2} \right) v(t). \]
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The reciprocal of the replacement impedance is \( \frac{i(t)}{v(t)} \), hence

\[ \frac{1}{Z_{\text{par}}} = \frac{1}{Z_1} + \frac{1}{Z_2} \]
Composition in parallel

Exercise

What is the replacement capacitance $C'$ of two parallel capacitors?

The reciprocal of the replacement impedance is $\frac{i(t)}{v(t)}$, hence

$$\frac{1}{Z_{\text{par}}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$
Composition in parallel

\[ i(t) = i_1(t) + i_2(t) = Z_1 v(t) + Z_2 v(t) = (1/Z_1 + 1/Z_2) v(t) \]

Exercise
What is the replacement capacitance \( C \) of two parallel capacitors?

Answer

\[ j\omega C = \frac{1}{1/j\omega C} = \frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \]

\[ = \frac{1}{1/j\omega C_1} + \frac{1}{1/j\omega C_2} = j\omega C_1 + j\omega C_2 \]

\[ = j\omega (C_1 + C_2), \]

hence \( C = C_1 + C_2 \).
The for the impedance of an inductor $L$ parallel to a capacitor $C$ we have

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$$\frac{1}{Z} = \frac{1}{j\omega L} + \frac{1}{1/j\omega C} = \frac{1}{j\omega L} + j\omega C = \frac{1 - \omega^2 LC}{j\omega L}.$$
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The impedance of the circuit is

$$Z = \frac{j\omega L}{1 - \omega^2 LC}.$$
The resonance frequency

\[ L \rightarrow C \quad \rightarrow \quad \text{Impedance:} \quad Z = \frac{j\omega L}{1 - \omega^2 LC} \]
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The impedance becomes very large if \( \omega^2 \approx \frac{1}{LC} \).
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The frequency $\omega_{\text{res}} = \frac{1}{\sqrt{LC}}$ is called the **resonance frequency**.

$$Z = \frac{j\omega L}{1 - \omega^2 LC}$$
Answer the following questions for the circuits (1), (2) and (3).

(a) What is the replacement impedance $Z$? Write $Z$ in canonical form.

(b) What happens with $|Z|$ for high frequencies ($\omega \to \infty$)?

(c) What happens with $|Z|$ for low frequencies ($\omega \to 0$)?